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Critical Behavior in a Quasi D Dimensional Spin Model[1]

Erhard Seiler[2] and Karim Yildirim[2]

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We study a classical spin model (more precisely a class of models) with $O(N)$ symmetry that can be viewed as a simplified D dimensional lattice model. It is equivalent to a non-translationinvariant one dimensional model and contains the dimensionality D as a parameter that need not be an integer. The critical dimension turns out to be 2, just as in the usual translation invariant models. We study the phase structure, critical phenomena and spontaneous symmetry breaking. Furthermore we compute the perturbation expansion to low order with various boundary conditions. In our simplified models a number of questions can be answered that remain controversial in the translation invariant models, such as the asymptoticity of the perturbation expansion and the role of super-instantons. We find that perturbation theory produces the right asymptotic expansion in dimension $D \leqslant 2$ only with special boundary conditions. Finally the model allows a test of the percolation ideas of Patrascioiu and Seiler.

KEY WORDS: Critical behavior; classical spin system; superinstantons; perturbation theory; percolation.

1. INTRODUCTION

In their well-known paper on the Mermin–Wagner theorem,[(1)] Dobrushin and Shlosman considered in a side remark a model that is the prototype of the model studied in this paper. Their purpose was to illustrate the importance of the smoothness of the interaction for the question of symmetry breaking. Here we take advantage of the fact that this model on the one hand has almost the simplicity of a one-dimensional model but on the other hand has a tunable parameter D playing the role of dimension. For

[1] Dedicated to the memory of Adrian Patrascioiu

[2] Max–Planck-Institut für Physik, Werner–Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany; e-mail: ehs@mppmu.mpg.de

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integer values of D, the model can actually be implemented as a model on the lattice $\mathbb{Z}^D$ with nontranslational Hamiltonian.

Such a model was also studied in ref. 2 in order to verify that the smoothness restrictions of ref. 1 can be relaxed and are only needed in a neighborhood of the minima of the Hamiltonion. Finally, in Georgii's book[3] a similar non-translation invariant chain, but with Ising spins is analyzed.

Here we use this type of model as a laboratory to test various ideas proposed by Patrascioiu and the first named author in their quest to prove that the conventional distinction between abelian and nonabelian is unjustified. The plan of the paper is as follows: After giving the definitions of the model and the various boundary conditions used, in Section 3 we use the full, nonperturbative solution of the model to study the phase structure as a function of the "dimension" D. It turns out that the critical dimension is still $D=2$; for $D \leqslant 2$ the model does not show spontaneous magnetization or phase coexistence, whereas for $D>2$ is does. For $D=2$ the model (without external magnetic field) and does not show asymptotic freedom.

In Section 4 we contrast those nonperturbative results with the results of (low order) perturbation theory (PT). We find that for zero magnetic field PT at the level of one loop already becomes dependent on the boundary conditions (b.c.) used for all $D \leqslant 2$ and therefore in general does not yield the correct asymptotic expansion of the model. The analogous result for the $1D$ model is well known (refs. 4 and 5). In this simple class of models it is, however, easy to find b.c. in which local observables are independent of the size L of the system and hence PT of the finite system does give the correct asymptotic expansion. Such b.c. exist in principle also for the full translation invariant models. Formally they arise by integrating out all the variables outside a box of suitable size, more precisely these b.c. are obtained using the Dobrushin–Lanford–Ruelle (DLR) equations.[6, 7] In our casefor $1 \leqslant D \leqslant 2$ those DLR boundary conditions turns out to be simply the standard free b.c. Whereas the nonperturbative results are rather easily obtained for our model, PT turns out to be harder to compute; for this reason we limit ourselves to one loop. This is sufficient, however, for seeing all the effects and subtleties we are interested in.

Finally, in Section 5, we discuss some percolation properties of our model. In particular we test the ideas of refs. 8 and 9 on the percolation properties of various sets defined by spins pointing in certain subsets of the spheres S_{N-1}. We find that in $1 \leqslant D \leqslant 2$ we are generally in the situation of "critical percolation," as suggested by refs. 8 and 9.

We should stress that our analysis shows that there is no qualitative difference between the abelian case $N=2$ and the nonabelian one ($N>2$). In that sense it lends support to the "heretical" scenario of Patrascioiu and

Seiler that predicts the existence of a "soft" phase in all $2D$ $O(N)$ models. Sceptics might still argue, however, that the model is more ordered than the standard translation invariant ones and that this is the reason for the existence of the soft phase.

In this paper we are to a large extent presenting results of ref. 10, with emphasis on the physical interpretation. For more mathematical and computational details we refer the reader to that work.

2. THE MODEL

2.1. Definition

To motivate the model, we start with a general class of classical $O(N)$ spin models defined on lattices $\mathbb{Z}^D$, but with link dependent couplings. To each lattice site $x \in \mathbb{Z}^D$ we associate a classical $O(N)$ spin, i.e.,

$$x \to \mathbf{s}(x), \qquad \mathbf{s}(x) \in \mathbb{R}^N, \qquad \|\mathbf{s}(x)\| = 1. \tag{2.1}$$

The Hamiltonian is of the form

$$H = -\sum_{\langle xy \rangle} \beta_{xy} \mathbf{s}(x) \cdot \mathbf{s}(y) - h \sum_x s_N(x) \tag{2.2}$$

where $s_N(j) = \mathbf{s}(j) \cdot \mathbf{e}_N$ and $\mathbf{e}_N$ is the unit vector in $\mathbb{R}^N$ pointing in the Nth direction; the sum is over nearest neighbors and h represents an external magnetic field; the dot denotes the standard euclidean scalar product in $\mathbb{R}^N$.

We now choose an origin and surround it with a family of concentric quadratic (cubic, hypercubic) "shells" (see Fig. 1). We "freeze" all the links (nearest neighbor pairs) sitting inside one of the shells by sending the corresponding $\beta_{xy} \to \infty$, thereby forcing all spins within such a shell to be equal. All other couplings β_{xy} are set equal to a constant β.

We can therefore identify all the points within a "shell;" thus the resulting model can be equivalently described as a non translation invariant semi-infinite spin chain formally defined by the Hamiltonian

$$H = -\sum_{j=1}^{\infty} (b_j \mathbf{s}(j) \cdot \mathbf{s}(j+1) - h_j s_N(j)) \tag{2.3}$$

where

$$b_j = 2D(2j-1)^{D-1}, \qquad j = 1, 2, 3,... \tag{2.4}$$

$$h_j = h g_j \tag{2.5}$$

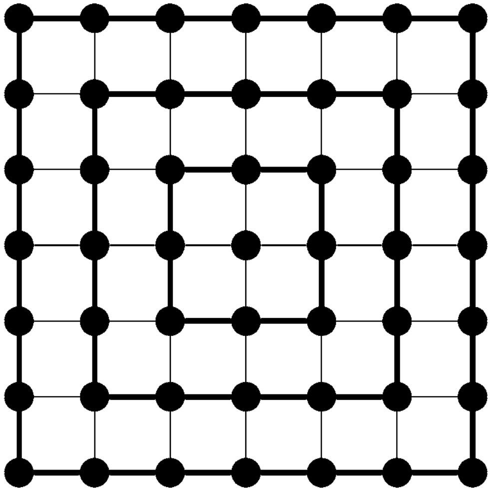

Fig. 1. Scheme of the lattice for $D=2$; on the thick lines we send $\beta_{xy}\to\infty$.

with

$$g_j=[(2j-1)^D-(2j-3)^D], \qquad j=2, 3, 4,... \tag{2.6}$$

and

$$g_1=1. \tag{2.7}$$

Note that for $j\to\infty$ both b_j and g_j behave asymptotically like $2D(2j)^{D-1}$.

As usual, we will have to study first finite chains of length L and then take the thermodynamic limit $L\to\infty$. This will require imposing certain b.c. at the end of the chain. In addition we will also generalize the model by introducing similar b.c. at the beginning of the chain. The form of the Hamiltonian given above corresponds to free b.c. at the beginning of the chain, which is natural from the point of view of the D-dimensional lattice, but we will also be interested in posing Dirichlet b.c. at the origin, which in combination with Dirichlet b.c. at $j=L$ will correspond to the superinstanton b.c. introduced in ref. 11. A class of b.c. that allows to interpolate between free b.c. and Dirichlet b.c. is given by the following finite volume Gibbs measures:

$$d\mu_L=\frac{1}{Z_L}\,e^{-\beta H_L}\prod_{j=1}^{L} d\nu(s(j)) \tag{2.8}$$

where dv is the $O(N)$ invariant probablity measure on S_{N-1} and

$$H_L = -\sum_{j=1}^{L-1} b_j \mathbf{s}(j) \cdot \mathbf{s}(j+1) - \sum_{j=1}^{L} h_j s_N(j) - a s_N(1) - b s_N(L). \qquad (2.9)$$

Free b.c. at $j=1$ ($j=L$) are given by putting $a=0$ ($b=0$); Dirichlet b.c. at $j=1$ ($j=L$) are obtained by taking the limit $a \to \infty$ ($b \to \infty$). Putting $b = b_L$, however, also corresponds to Dirichlet b.c., but at $j = L+1$ for a chain of length $L+1$. Likewise $a=1$ can be interpreted as Dirichlet b.c. on an extended chain.

We will also make use of more general b.c. at $j=L$ which are obtained by replacing the factor $\exp(\beta b s_N(L))\, dv(\mathbf{s}(L))$ in the Gibbs measure for a finite chain with a general positive measure written formally (in distributional notation) as $\tilde{\psi}(\mathbf{s}(L))\, dv(\mathbf{s}(L))$. We may even let $\tilde{\psi}$ depend on the length L of the chain; in particular this is necessary if we want to describe Dirichlet b.c. at $j = L+1$ by choosing $b = b_L$, as described above.

The model we have defined contains the dimension D as a parameter which no longer has to be an integer (it could even be chosen complex). In the following we will treat D as a real parameter $\geqslant 1$ and we will use almost exclusively the representation of the model on a chain.

Let us note one crucial fact which turns out to be responsible for the dependence of the properties of the model on the "quasidimension" D:

Proposition 2.1. For $1 \leqslant D \leqslant 2$ and any $k > 0$ $\sum_{j=k}^{L} b_j^{-1} \to \infty$ for $L \to \infty$, whereas for $D > 2$ $\sum_{j=k}^{L} b_j^{-1}$ converges to a finite limit. More precisely, for $D < 2$ $\sum_{j=k}^{L} b_j^{-1} = O(k^{2-D})$, for $D = 2$ it is $O(\ln k)$, and for $D > 2$ it approaches its limit like k^{2-D}.

The proof is a straightforward consequence of the asymptotics of the b_j.

2.2. Thermodynamic Quantities

To define thermodynamic quantities like free energy density, magnetization or susceptibility, we first note that the volume of the chain of length L is naturally defined as

$$V_L = (2L+1)^D. \qquad (2.10)$$

The free energy density then becomes

$$f_L = -\frac{1}{\beta V_L} \ln Z_L \qquad (2.11)$$

and the magnetization we define as usual as

$$M_L = -\frac{\partial f_L}{\partial h}. \tag{2.12}$$

Thus

$$M_L = \frac{1}{V_L} \sum_{j=1}^{L} g_j \langle s_N(j) \rangle. \tag{2.13}$$

Likewise we define the (longitudinal) susceptibility as

$$\chi_L = \frac{\partial M_L}{\partial h} = -\frac{\partial^2 f}{\partial h^2} \tag{2.14}$$

which means

$$\chi_L = \frac{\beta}{V_L} \sum_{j,l=1}^{L} g_j g_l [\langle s_N(j)\, s_N(l) \rangle - \langle s_N(j) \rangle \langle s_N(l) \rangle]. \tag{2.15}$$

All these definitions are analogous to those in translation invariant systems. Of course we are mostly interested in the thermodynamic limit $L \to \infty$. We use the definitions (2.13) and (2.15) to define the spontaneous magnetization and the susceptibility also for the thermodynamic limit at $h = 0$.

2.3. Superinstantons

For $D \leqslant 2$ the model has superinstantons, just like the translation invariant model.[11] These are configurations of arbitrarily low energy which are disordering the system and are responsible for the absence of spontaneous symmetry breaking (Mermin–Wagner theorem[12]).

More concretely, these configurations are obtained by imposing b.c. at $j = i$ and $j = k$ such that $\mathbf{s}(i) \cdot \mathbf{s}(k) = \arccos(\alpha) < 1$ and minimizing the energy under those b.c. The existence of such a minimizing configuration is obvious, because the energy is a continuous function on a compact set. It is clear that in this minimal configuration the spin will vary on a great circle in S_{N-1}, so we can describe the configuration by the angle

$$\phi_j = \arccos(\mathbf{s}(j) \cdot \mathbf{s}(j+1)). \tag{2.16}$$

The quantitiy to be minimized is then

$$E(i, k) = -\sum_{j=i}^{k-1} b_j(\cos(\phi_j) - 1) \tag{2.17}$$

under the condition that

$$\sum_{j=i}^{k-1} \phi_j = \alpha. \tag{2.18}$$

Using a Lagrange multiplicator λ one obtains therefore the equations

$$\lambda - b_j \sin(\phi_j) = 0 \tag{2.19}$$

which has the solutions

$$\phi_j = \arcsin\left(\frac{\lambda}{b_j}\right) \tag{2.20}$$

and yields for the minimizing configuration

$$E_{\text{s.i.}}(i, k) = \sum_{j=i}^{k-1} b_j \left(1 - \sqrt{1 - \frac{\lambda^2}{b_j^2}}\right) \tag{2.21}$$

with

$$\sum_{j=i}^{k-1} \arcsin\left(\frac{\lambda}{b_j}\right) = \alpha. \tag{2.22}$$

We will now derive upper and lower bounds for $E_{\text{s.i.}}(i, k)$. Using

$$1 - x \leqslant \sqrt{1 - x} \leqslant 1 - \frac{x}{2} \qquad \text{for} \quad 0 \leqslant x \leqslant 1 \tag{2.23}$$

we obtain

$$\frac{1}{2} \sum_{j=i}^{k-1} \frac{\lambda^2}{b_j} \leqslant E_{\text{s.i.}}(i, k) \leqslant \sum_{j=i}^{k-1} \frac{\lambda^2}{b_j}. \tag{2.24}$$

Bounds for the Lagrange multiplier λ are obtained using

$$x \leqslant \arcsin(x) \leqslant 2x \qquad \text{for} \quad 0 \leqslant x \leqslant 1 \tag{2.25}$$

from which we obtain, using Eq. (2.19)

$$\lambda \sum_{j=i}^{k-1} b_j^{-1} \leqslant \alpha \leqslant 2\lambda \sum_{j=i}^{k-1} b_j^{-1} \tag{2.26}$$

or

$$\frac{\alpha}{2 \sum_{j=i}^{k-1} b_j^{-1}} \leqslant \lambda \leqslant \frac{\alpha}{\sum_{j=i}^{k-1} b_j^{-1}}. \tag{2.27}$$

Combining this with Eq. (2.24) we finally obtain

$$\frac{\alpha^2}{8 \sum_{j=i}^{k-1} b_j^{-1}} \leqslant E_{\text{s.i.}}(i, k) \leqslant \frac{\alpha^2}{\sum_{j=i}^{k-1} b_j^{-1}}. \tag{2.28}$$

Here we can see again the distinction between low ($D \leqslant 2$) and high ($D > 2$) quasidimension:

For $D \leqslant 2$

$$\lim_{k \to \infty} E_{\text{s.i.}}(i, k) = 0 \tag{2.29}$$

whereas for $D > 2$ $E_{\text{s.i.}}(i, k)$ is uniformly in k bounded away from 0:

$$E_{\text{s.i.}}(i, k) \geqslant \frac{\alpha^2}{8 \sum_{j=i}^{\infty} b_j^{-1}} > 0. \tag{2.30}$$

The fact that the superinstantons in $D \leqslant 2$ cost arbitrarily little energy is responsible for the fact that the system has no long range order (just as the translation invariant models are according to the Mermin–Wagner theorem(12)). This will be proven in the next section.

Due to the freezing of all the spins within one "layer," the model does not have instanton-like configurations.

3. NONPERTURBATIVE ANALYSIS

3.1. Transfer Matrices

Because our model only couples nearest neighbors, it can be described easily in terms of transfer operators, which are of course site dependent. These transfer "matrices" are trace class operators on the Hilbert space

$$\mathscr{H} = L^2(S_{N-1}, dv), \tag{3.1}$$

which we normalize such that the largest eigenvalue equals 1. In slight abuse of notation, we will use the same symbol for the operators and their integral kernels.

The normalized transfer matrix from site j to site $j+1$ is

$$\tilde{\mathscr{T}}_j(\mathbf{s}(j), \mathbf{s}(j+1)) = \frac{1}{z_j} \exp[\beta(b_j \mathbf{s}(j) \cdot \mathbf{s}(j+1) + h_j s_N(j))] \tag{3.2}$$

with z_j chosen such that

$$\|\tilde{\mathscr{T}}_j\| = 1. \tag{3.3}$$

These transfer operators are not self-adjoint, but it is easy to transform them by a similarity transformation with a bounded operator into the self-adjoint operators with integral kernel

$$\mathscr{T}_j(\mathbf{s}(j), \mathbf{s}(j+1)) = \frac{1}{z_j} \exp\left[\beta(b_j \mathbf{s}(j) \cdot \mathbf{s}(j+1)) + \frac{h_j}{2} (s_N(j) + s_N(j+1)) \right]. \tag{3.4}$$

These operators are in fact positive, as can be seen easily be expanding $\exp[\mathbf{s}(j) \cdot \mathbf{s}(j+1)]$.

Since the transfer operators have also positive integral kernels, by a trivial change of normalization it is possible to interprete them as transition probabilities and the whole system as a Markov chain. This point of view will, however, not play a great role in this paper.

We introduce a shorthand for products of transfer matrices:

$$\mathscr{T}_{jk} \equiv \prod_{j=1}^{k-1} \mathscr{T}_j \tag{3.5}$$

and define

$$\beta_j \equiv \beta b_j. \tag{3.6}$$

Using the transfer matrices, it is possible to rewrite for instance the expectation value of a spin component at site k

$$\langle s_a(k) \rangle_L \equiv \int d\mu_L \; s_a(i) \tag{3.7}$$

in the form

$$\langle s_a(i) \rangle_L = \frac{(\psi, \mathscr{T}_{1i} s_a \mathscr{T}_{iL} \tilde{\psi})}{(\psi, \mathscr{T}_{1L} \tilde{\psi})} \tag{3.8}$$

where we denote by $(.,.)$ the scalar product in $\mathscr{H}$. ψ and $\tilde{\psi}$ are suitable vectors in $\mathscr{H} = L^2(S_{N-1})$ describing the b.c. (actually we may even replace them by general positive measures on S_{N-1}). The spin variable s_a appearing on the right hand side of Eq. (3.8) is to be interpreted as a multiplication operator on $L^2(S_{N-1})$.

The spin-spin two point function

$$\langle s_a(i)\, s_b(k)\rangle_L \equiv \int d\mu_L\, s_a(i)\, s_b(k) \tag{3.9}$$

for $i \leqslant k$ can similarly be expressed as

$$\langle s_a(i)\, s_b(k)\rangle_L = \frac{(\psi, \mathscr{T}_{1i}\mathbf{s}_a\mathscr{T}_{ik}\mathbf{s}_b\mathscr{T}_{kL}\tilde{\psi})}{(\psi, \mathscr{T}_{1L}\tilde{\psi})}. \tag{3.10}$$

For a nonvanishing magnetic field we do not expect any interesting phenomena; the transfer matrices will force the spins "at infinity" to be aligned with the direction of the magnetic field $\mathbf{e}_N$, therefore one expects a unique thermodynamic limit, independent of the b.c. imposed at L.

For vanishing magnetic field the situation is more interesting; therefore from now on we will assume $h = 0$. To analyze the situation, we need some preparation (see for instance refs. 13 and 14).

The Hilbert space $\mathscr{H} = L^2(S_{N-1}, dv)$ can be decomposed into the eigenspaces $\mathscr{H}_l$ of the Laplace–Beltrami operator Δ_{LB} on the sphere S_{N-1}:

$$\mathscr{H} = \bigoplus_{l=0}^{\infty} \mathscr{H}_l. \tag{3.11}$$

The projections $\mathscr{P}_l$ onto the eigenspaces are integral operators which for $N > 2$ can be expressed in terms of the Gegenbauer polynomials C_l (see refs. 15 and 16)

$$\begin{aligned} C_l^{\frac{N}{2}-1}(x) &= \frac{1}{\Gamma(\frac{N}{2}-1)} \sum_{m=0}^{[\frac{l}{2}]} (-1)^m \frac{\Gamma(\frac{N}{2}-1+l-m)}{m!\,(l-2m)!} (2x)^{l-2m}, \qquad N > 2, \\ C_l^0(x) &= \sum_{m=0}^{[\frac{l}{2}]} (-1)^m \frac{\Gamma(l-m)}{\Gamma(m+1)\,\Gamma(l-2m+1)} (2x)^{l-2m}, \qquad l \neq 0 \end{aligned} \tag{3.12}$$

as follows:

$$\mathscr{P}_l(\mathbf{s}, \mathbf{s}') = \frac{2l+N-2}{N-2}\, C_l^{\frac{N}{2}-1}(s\cdot s'). \tag{3.13}$$

For $N = 2$ the Gegenbauer polynomials have to be replaced by the Chebyshev polynomials of the first kind:

$$T_o(x) = 1 \tag{3.14}$$

$$T_l(x) = \lim_{N \to 2} \frac{1}{N-2} C_l^{\frac{N}{2}-1}(x) \qquad \text{for} \quad l > 1. \tag{3.15}$$

In the following we will write the equations in the form valid for $N > 2$, involving the Gegenbauer polynomials, with the understanding that for $N = 2$ the analogous formulae involving the Chebyshev polynomials hold.

Due to the $O(N)$ invariance of the transfer matrices for $h = 0$, all $\mathcal{T}_j$ commute with those projections, and the spaces $\mathcal{H}_l$ are also simultaneous eigenspaces of all $\mathcal{T}_j$. Because the Gegenbauer (Chebyshev) polynomials form a complete orthogonal set on the interval $[-1, 1]$, the integral kernel of the transfer matrix can be expanded in the sense of $L^2([-1, 1], (1-x^2)^{\frac{N-2}{2}} dx)$ as

$$\mathcal{T}_j(\mathbf{s}, \mathbf{s}') = \sum_{l=0}^{\infty} c_l C_l^{N/2-1}(\mathbf{s} \cdot \mathbf{s}'). \tag{3.16}$$

From this we see that each subspace $\mathcal{H}_l$ is an eigenspace with eigenvalue λ_l of $\mathcal{T}_j$. The eigenvalues λ_l are therefore given by

$$\begin{aligned} \lambda_l(\beta_j) &= \frac{\operatorname{tr} \mathcal{T}_j \mathcal{P}_l}{\operatorname{tr} \mathcal{P}_l} = \frac{1}{z_j} \int_{-1}^{1} dx \exp(\beta_j x)(1-x^2)^{\frac{N-3}{2}} \frac{C_l^{n/2-1}(x)}{C_l^{n/2-1}(1)} \\ &= \frac{1}{z_j} \sqrt{\pi}\, \Gamma\left(\frac{N-1}{2}\right) \left(\frac{2}{\beta_j}\right)^{\frac{N-2}{2}} I_{l+\frac{N}{2}-1}(\beta_j) \end{aligned} \tag{3.17}$$

where $I_r(.)$ is the modified Bessel function (ref. 15). From the fact that the $\mathcal{T}_j$ are positivity improving operators (see ref. 17) it follows that

$$0 < \lambda_l(\beta_j) < \lambda_o(\beta_j) = 1. \tag{3.18}$$

The normalization $\lambda_o(\beta_j) = 1$ allows to determine z_j explicitly and we get for the eigenvalues finally

$$\lambda_l(\beta_j) = \frac{I_{l+\frac{N}{2}-1}(\beta_j)}{I_{\frac{N}{2}-1}(\beta_j)}. \tag{3.19}$$

3.2. Thermodynamic Limit

In this subsection we will discuss explicitly the thermodynamic limit for the one and two point functions of the spins. The generalization to higher n-point functions is straightforward in principle.

To take the thermodynamic limit we need some information about the limit of the product of a large number of transfer matrices. Since for $h=0$ all transfer matrices commute and have the same eigenprojections $\mathscr{P}_l$, we obtain

$$\mathscr{T}_{ik} = \sum_{l=0}^{\infty} \prod_{j=i}^{k-1} \lambda_l(\beta_j)\, \mathscr{P}_l. \tag{3.20}$$

So we have to study the behavior of the products $\prod_{j=i}^{k-1} \lambda_l(\beta_j)$ for large k. For this purpose we can use some results of refs. 14 and 13, where we analyzed the asymptotic behavior of such eigenvalues (actually for more general forms of the Hamiltonian). The main result is

$$\ln \lambda_l(\beta_j) = 1 - \frac{l(l+N-2)}{2\beta_j} + O(\beta_j^{-2}). \tag{3.21}$$

We define for $i \leqslant k$

$$\Lambda_l(i, k) = \prod_{j=i}^{k-1} \lambda_l(\beta_j). \tag{3.22}$$

Using (3.21) and Proposition 2.1 we find the asymptotic behavior of $\ln \Lambda_l(1, k)$:

$$\ln \Lambda_l(1, k) = -\frac{l(l+N-2)}{2\beta} \frac{1}{2^D D(2-D)} k^{2-D} + O\left(\frac{k^{2(2-D)}}{\beta^2}\right) \quad \text{for } 1 \leqslant D < 2, \tag{3.23}$$

$$\ln \Lambda_l(1, k) = -\frac{l(l+N-2)}{2\beta} \frac{1}{8} \ln k + O(\beta^{-2}) \qquad \text{for} \quad D = 2 \tag{3.24}$$

and

$$\ln \Lambda_l(k, \infty) = -\frac{l(l+N-2)}{2\beta} \frac{1}{2^D D(D-2)} k^{2-D} + O\left(\frac{k^{2(2-D)}}{\beta^2}\right) \qquad \text{for} \quad D > 2. \tag{3.25}$$

Using this, we obtain easily

Proposition 3.1. For $D \leqslant 2$ $\mathcal{T}_{i\infty} \equiv s\text{-lim}_{k\to\infty} \mathcal{T}_{ik} = \mathcal{P}_0$. For $D > 2$ $\mathcal{T}_{i\infty}$ exists but is strictly positive; $s\text{-lim}_{i\to\infty} \mathcal{T}_{i\infty} = \mathbb{1}$.

Proof. To prove strong convergence, it suffices by a 3ϵ argument to prove convergence of all the eigenvalues. We now treat three cases separately:

(1) $D=1$: All $\mathcal{T}_j$ are equal and the result follows trivally from the fact that for $l \geqslant 1$ $\lambda_l < 1$

(2) $1 < D \leqslant 2$: The assertion follows from the fact that for all $l \geqslant 1$

$$\lim_{k\to\infty} \Lambda_l(1, k) = 0. \tag{3.26}$$

To see this, use

$$\ln \prod_{j=i}^{k} \lambda_l(\beta_j) \leqslant \sum_{j=i}^{k} (\lambda_l(\beta_j) - 1) = -\sum_{j=i}^{k} \left[\frac{l(l+N-2)}{2\beta_j} + O(\beta_j^{-2}) \right]. \tag{3.27}$$

According to Proposition 2.1 the first sum diverges to $-\infty$; more precisely, it behaves like $-(k)^{2-D}$ for $1 < D < 2$ and like $\ln k$ for $D=2$. The sum of the correction terms diverges for $1 < D < 3/2$ at most like k^{3-2D}, for $D = 3/2$ like $\ln k$ and converges for $D > 3/2$. In any case the asymptotic behavior for large k is determined by the leading term; so the product of the eigenvalues diverges to zero and the assertion follows.

(3) $D > 2$: The product of the eigenvalues $\Lambda_l(k, \infty)$ converges absolutely to a nonzero value because

$$\sum_{j=i}^{k} |(\lambda_l(\beta_j) - 1)| \tag{3.28}$$

converges for $k \to \infty$. ∎

One and two point functions simplify in the thermodynamic limit, provided either $1 \leqslant D \leqslant 2$ or we have free b.c. at 1 (i.e., $a=1$. Free b.c. at 1 seem most natural anyway from the point of view of the D dimensional lattice; if we consider $O(N)$ invariant observables, however, it is just as legitimate to choose $a \neq 0$, since in the thermodynamic limit this only means fixing the global $O(N)$ invariance.

We first discuss the case $1 \leqslant D \leqslant 2$ and $a=0$. In this case the Hilbert space vector ψ in the expression (3.8) becomes proportional to the "ground state" ψ_o, which is the function identically equal to 1 on S_{N-1}. ψ_0 spans the range of $\mathscr{P}_0$; it is invariant under all $\mathscr{T}_j$ and therefore we obtain

$$\langle s_a(i)\rangle_L = \frac{(\psi_0, s_a \mathscr{T}_{iL} \tilde{\psi})}{(\psi_0, \tilde{\psi})} \tag{3.29}$$

and

$$\langle s_a(i)\, s_b(k)\rangle_L = \frac{(\psi_0, s_a \mathscr{T}_{ik} s_b \mathscr{T}_{kL} \tilde{\psi})}{(\psi_0, \tilde{\psi})}. \tag{3.30}$$

Next we use the fact that s_a maps the subspace $\mathscr{H}_0$ into the subspace $\mathscr{H}_1$, to conclude (assuming $i \leqslant k$)

$$\langle s_a(i)\rangle_L = \frac{(\psi_0, s_a \tilde{\psi})}{(\psi_0, \tilde{\psi})}\, \Lambda_1(i, L) \tag{3.31}$$

and

$$\langle s_a(i)\, s_b(k)\rangle_L = \frac{(\psi_0, s_a s_b \mathscr{T}_{kL} \tilde{\psi})}{(\psi_0, \tilde{\psi})}\, \Lambda_1(i, k). \tag{3.32}$$

Now we take the thermodynamic limit, using Proposition 3.1. We obtain for $1 \leqslant D \leqslant 2$:

$$\langle s_a(i)\rangle_\infty = 0 \tag{3.33}$$

and

$$\langle s_a(i)\, s_b(k)\rangle_\infty = (\psi_0, s_a s_b \psi_0)\, \Lambda_1(i, k). \tag{3.34}$$

By $O(N)$ invariance it is easy to see that

$$(\psi_0, s_a s_b \psi_0) = \frac{1}{N}\, \delta_{ab}. \tag{3.35}$$

More generally, by the same reasoning, if F is a bounded measurable function of finitely many spins $s(i_1)$,..., $s(i_k)$, and $\bar{F}$ its $O(N)$ average,

$$\langle F\rangle_\infty = \langle \bar{F}\rangle_\infty. \tag{3.36}$$

Finally let us discuss what happens for $1 \leqslant D \leqslant 2$ and $a \neq 0$, provided we consider $O(N)$ invariant observables. So let $\bar{F}$ now be an $O(N)$ invariant bounded measurable function of finitely many spins. In this case the Hilbert space vector ψ in the expression (3.8) becomes a function $\psi(a)$ with $(\psi(a), \psi_0) \neq 0$. But in the thermodynamic limit, again using Proposition 3.1, we obtain

$$\langle \bar{F} \rangle_{\infty, a, b} = \frac{(\psi(a), \bar{F}\psi_0)}{(\psi(a), \psi_0)} = (\psi_0, \bar{F}\psi_0), \tag{3.37}$$

which is independent of a and b.

We summarize what we have found for "low" dimension in

Theorem 3.2. For $1 \leqslant D \leqslant 2$ and $a = 0$ (free b.c. at 1) in the thermodynamic limit, irrespective of the b.c. at L (i.e., b) the following holds:

- $\langle s_a(i) \rangle_\infty = 0$
- $\langle s_a(i)\, s_b(k) \rangle_\infty = \frac{1}{N}\, \delta_{ab} \prod_{j=i}^{k-1} \lambda_1(\beta_j) = \frac{1}{N}\, \delta_{ab} \Lambda_1(i, k)$.

In addition for any bounded measurable function F of finitely many spins, and $\bar{F}$ its $O(N)$ average, we have

$$\langle F \rangle_\infty = \langle \bar{F} \rangle_\infty. \tag{3.38}$$

Furthermore for arbitrary a $\langle \bar{F} \rangle_\infty$ is independent of a and b. Finally for $k \to \infty$ we have the asymptotic behavior

$$\langle s_a(i)\, s_b(k) \rangle_\infty \sim \delta_{ab} \frac{1}{N} \exp\left(-\frac{N-1}{2D2^D \beta(2-D)} k^{2-D} \right) \qquad (1 \leqslant D < 2) \tag{3.39}$$

and

$$\langle s_a(i)\, s_b(k) \rangle_\infty \sim \delta_{ab} \frac{1}{N} \exp\left(-\frac{N-1}{16\beta} \ln k \right) \qquad (D = 2). \tag{3.40}$$

For $D > 2$ the situation is different and a little more involved. We have

$$\langle s_a(i) \rangle_\infty = \frac{(\psi_0, s_a \tilde{\psi})}{(\psi_0, \tilde{\psi})} \Lambda_1(i, \infty) \tag{3.41}$$

and

$$\langle s_a(i)\, s_b(k) \rangle_\infty = \frac{(\psi_0, s_a s_b \mathscr{T}_{k\infty} \tilde{\psi})}{(\psi_0, \tilde{\psi})} \Lambda_1(i, k). \tag{3.42}$$

If we choose Dirichlet b.c. by sending $\tilde{\psi}$ to a delta function $\delta_{\mathbf{e}_N}$ concentrated at $\mathbf{s} = \mathbf{e}_N$, the one point function simplifies to

$$\langle s_a(i)\rangle_{\infty,\,\mathrm{Dir}} = \delta_{aN} \Lambda_1(i, \infty). \tag{3.43}$$

We can analyze the 2 point function further by using the fact that only the subspaces $\mathscr{H}_0$ and $\mathscr{H}_2$ contribute here; generally we have

$$(\psi_0, s_a s_b \phi) = \frac{\delta ab}{N} (\psi_0, \mathbf{s}^2 \mathscr{P}_0 \phi) + \left(\psi_0, \left(s_a s_b - \frac{\delta_{ab}}{N} \mathbf{s}^2\right) \mathscr{P}_2 \phi\right) \tag{3.44}$$

for any $\phi \in \mathscr{H}$. Inserting $\phi = \mathscr{T}_{k\infty} \tilde{\psi}$ we obtain

$$\begin{aligned}\langle s_a(i)\, s_b(k)\rangle_\infty &= \delta_{ab} \frac{1}{N} \frac{(\psi_0, \mathbf{s}^2 \tilde{\psi})}{(\psi_0, \tilde{\psi})} \Lambda_1(i, k) \\ &\quad + \frac{(\psi_0, (s_a s_b - \delta_{ab} \frac{1}{N} \mathbf{s}^2)\, \tilde{\psi})}{(\psi_0, \tilde{\psi})} \Lambda_1(i, k)\, \Lambda_2(k, \infty).\end{aligned} \tag{3.45}$$

Using the fact that

$$\frac{(\psi_0, \mathbf{s}^2 \tilde{\psi})}{(\psi_0, \tilde{\psi})} = 1 \tag{3.46}$$

and taking the limit $k \to \infty$ we obtain

$$\lim_{k \to \infty} \langle s_a(i)\, s_b(k)\rangle_\infty = \left[\delta_{ab} \frac{1}{N} + \left(\psi_0, \left(s_a s_b - \delta_{ab} \frac{1}{N} \mathbf{s}^2\right) \tilde{\psi}\right)\right] \Lambda_1(i, \infty). \tag{3.47}$$

This latter expression is in general nonzero, but for Dirichlet b.c. ($\tilde{\psi} = \delta_{\mathbf{e}_N}$ we can evaluate it further, because in this case

$$\left(\psi_0, \left(s_a s_b - \frac{\delta_{ab}}{N} \mathbf{s}^2\right) \tilde{\psi}\right) = \delta_{aN} \delta_{b1} - \delta_{ab} \frac{1}{N} \tag{3.48}$$

and we obtain

$$\lim_{k \to \infty} \langle s_a(i)\, s_b(k)\rangle_{\infty,\,\mathrm{Dir}} = \delta_{aN} \delta_{bN} \Lambda_1(i, \infty). \tag{3.49}$$

Comparing with the result (3.43) for the one point function we see that the latter limit equals

$$\lim_{k \to \infty} \langle s_a(i)\rangle_{\infty,\,\mathrm{Dir}} \langle s_a(k)\rangle_{\infty,\,\mathrm{Dir}}. \tag{3.50}$$

In other words, for Dirichlet b.c. at $L \to \infty$, the truncated two point function

$$\langle s_a(i); s_b(k) \rangle_{\infty,\,\mathrm{Dir}} \equiv \langle s_a(i)\, s_b(k) \rangle_{\infty,\,\mathrm{Dir}} - \langle s_a(i) \rangle_{\infty,\,\mathrm{Dir}} \langle s_b(k) \rangle_{\infty,\,\mathrm{Dir}} \tag{3.51}$$

goes to 0 for $k \to \infty$. In this sense we may interprete the state with Dirichlet b.c. at ∞ as a pure phase of the system. Note also that by (3.25) the limit is approached like k^{2-D}.

We can still work out the truncated two point function for Dirichlet b.c. at ∞ in a little more detail: from Eq. (3.45) we have

$$\begin{aligned}\langle s_a(i); s_b(k) \rangle_{\infty,\,\mathrm{Dir}} = {} & \delta_{ab} \frac{1}{N} \Lambda_1(i,k) + \left(\delta_{aN} \delta_{b_N} - \delta_{ab} \frac{1}{N} \right) \Lambda_1(i,k)\, \Lambda_2(k,\infty) \\ & - \delta_{aN} \delta_{b_N} \Lambda_1(i,\infty)\, \Lambda_1(k,\infty) \end{aligned} \tag{3.52}$$

which can be rewritten as

$$\begin{aligned} & \left(\delta_{aN} \delta_{b_N} - \delta_{ab} \frac{1}{N} \right) \Lambda_1(i,k)(\Lambda_2(k,\infty) - 1) \\ & + \delta_{aN} \delta_{b_N} (\Lambda_1(i,k) - \Lambda_1(i,\infty)\, \Lambda_1(k,\infty)). \end{aligned} \tag{3.53}$$

For $a = b \neq N$ this is manifestly positive. Furthermore, it is also straightforward to see that the $O(N)$ invariant truncated two point function is positive:

$$\langle \mathbf{s}(i); \mathbf{s}(k) \rangle > 0 \tag{3.54}$$

where the semicolon is meant to include the dot symbolizing the scalar product in $\mathbb{R}^N$.

Let us now summarize what has been found for $D > 2$ in

Theorem 3.3. For $D > 2$ and and $a = 0$ (free b.c. at 1) in the thermodynamic limit

- the one-point function is

$$\langle s_a(i) \rangle_\infty = \Lambda_1(i,\infty) \frac{(\psi_0, s_a \tilde{\psi})}{(\psi_0, \tilde{\psi})} \tag{3.55}$$

which is nonzero provided the scalar product in the numerator does not vanish. This is in particular the case if $s_a = s_N$ and the b.c. parameter $b > 0$.

- $\lim_{k\to\infty} \langle s_a(i)\, s_b(k)\rangle_\infty$ is generically nonzero.
- For Dirichlet b.c. ($b \to \infty$) the one point function is

$$\langle s_a(i)\rangle_{\infty,\,\mathrm{Dir}} = \delta_{aN} \Lambda_1(i,\infty); \tag{3.56}$$

furthermore the two point function has the cluster property

$$\lim_{k\to\infty} \langle s_a(i);\, s_b(k)\rangle_{\infty,\,\mathrm{Dir}} = 0. \tag{3.57}$$

The truncated two point function behaves like

$$\langle s_a(i);\, s_a(k)\rangle \equiv \langle s_a(i)\, s_b(k)\rangle_{\infty,\,\mathrm{Dir}} - \langle s_a(i)\rangle_{\infty,\,\mathrm{Dir}} \langle s_a(k)\rangle_{\infty,\,\mathrm{Dir}} = O(k^{2-D}). \tag{3.58}$$

From the two theorems we can easily derive results about the magnetization:

Corollary 3.4. For $1 \leqslant D \leqslant 2$ the spontaneous magnetization $M_\infty = \lim_{L\to\infty} M_L$ vanishes, irrespective of the b.c. For $D > 2$ the model has spontaneous magnetization with suitable b.c.; for Dirichlet b.c. $M_\infty = 1$.

Proof. The spontaneous magnetization M_∞ is, according to 2.13 a certain weighted average of the 1-point function $\langle s_N(i)\rangle_\infty$. The one point function converges for $i \to \infty$ to

$$\langle s_N(\infty)\rangle_\infty = \frac{(\psi_0,\, s_N \tilde{\psi})}{(\psi,\, \tilde{\psi})}. \tag{3.59}$$

Any average of a convergent sequence is equal to its limit, so we have

$$M_\infty = \langle s_N(\infty)\rangle_\infty. \tag{3.60}$$

For Dirichlet b.c., because $\tilde{\psi}$ becomes the distribution $\delta_{\mathbf{e}_N}$ concentrated at $\mathbf{e}_N$, the last quantity is equal to 1. ∎

The physical interpretation of the fact that the truncated two point function $\langle s_a(i)\, s_a(k)\rangle$ decays faster than any power of k for $1 \leqslant D \leqslant 2$ is that the model is in its high temperature phase for any value of β. Likewise the power-like decay in $D = 2$ means that the model is critical for any β. The power-like decay of $\langle s_a(i);\, s_a(k)\rangle$ for $D > 2$ expresses the presence of a Goldstone-like mode in the system.

The susceptibility is a little pathological in our model. As defined above, it diverges in general (except for $D = 1$) due to the freezing of the spins within one layer. Namely, if the truncated two point function is non-negative (as it is, according to the discussion above), we have

$$\chi_L = \frac{\beta}{V_L} \sum_{i,k=1}^{L} g_i g_k \langle s_N(i); s_N(k) \rangle \geqslant \sum_{i=1}^{L} g_i^2 \langle s_N(i)^2 \rangle = \frac{\beta}{NV_L} \sum_{i=1}^{L} g_i^2 \tag{3.61}$$

which diverges for $L \to \infty$.

3.3. Presence and Absence of Asymptotic Freedom

In this subsection we discuss the issue of asymptotic freedom using a definition of the Callan–Symanzik β-function β_{CS} due to Patrascioiu.[5] We use the truncated two point function

$$G_\beta(i,k) \equiv \langle \mathbf{s}(i); \mathbf{s}(k) \rangle \tag{3.62}$$

to define the following Renormalization Group (RG) invariant quantity:

$$F_\beta(i,k) \equiv \frac{G_\beta(2i,2k)}{G_\beta(i,k)}. \tag{3.63}$$

We rescale the lattice distances in F_β and ask how this can be compensated by a change of the coupling constant $g \equiv 1/\sqrt{\beta}$.

This compensation cannot be made exact by only changing β, but it works aymptotically in the limit of many iterations, or equivalently for the large distance asymptotics.

Since we have the asymptotic behavior (see (3.25))

$$\langle \mathbf{s}(i) \cdot \mathbf{s}(k) \rangle \sim \exp\left(\frac{N-1}{2^D D(D-2)\,\beta} (i^{2-D} - k^{2-D}) \right) \tag{3.64}$$

(for $k > i$), it is seen straightforwardly that a doubling of i and k can be compensated by replacing β with $2^{2-D}\beta$ or g with $2^{D/2-1}$. Of course there are corrections of order $1/\beta^k$ to the exponent and we will compute the first one of these corrections in the next section, but they do not affect the qualitative conclusions.

If we interpolate F_β to obtain a smooth function on $\mathbb{R}^2$ we can do this rescaling infinitesimally and find that F_β obeys, at least asympotically, the following RG equation:

$$(\partial_t + \beta_{CS}(g(t))\,\partial_g)\,F_\beta(e^t i, e^t k) = 0. \tag{3.65}$$

This equation should really be interpreted at the definition of the Callan–Symanzik β-function, i.e., we have to set

$$\beta_{CS}(g(t)) = -\frac{\partial_t F_\beta(e^t i, e^t k)}{\partial_g F_\beta(e^t i, e^t k)} \tag{3.66}$$

which yields, putting $t = 0$ and $g(0) = g$,

$$\beta_{CS}(g) = \frac{D-2}{2} g + O(g^2). \tag{3.67}$$

This result shows that for $D < 2$ the model is asymptotically free, whereas for $D = 2$ it is critical for any $\beta > 0$, i.e., we have a half-line of fixed points. This is true for any $N > 1$, i.e., there is no qualitative difference between the abelian and the nonabelian versions of the model.

For $D > 2$ the analysis is a little different, because we have to take the truncation into account. The asymptotic behavior of the truncated two point function is

$$\begin{aligned} \langle \mathbf{s}(i); \mathbf{s}(k) \rangle \sim \exp\left(-\frac{N-1}{2^D D(D-2)\,\beta} (i^{2-D} - k^{2-D}) \right) \\ - \exp\left(-\frac{N-1}{2^D D(D-2)\,\beta} (i^{2-D} + k^{2-D}) \right) \end{aligned} \tag{3.68}$$

i.e.,

$$\langle \mathbf{s}(i); \mathbf{s}(k) \rangle \sim -2\,\frac{N-1}{2^D D(D-2)\,\beta} k^{2-D}. \tag{3.69}$$

Forming the renormalization group invariant F_β we see that the dependence on β as well as the scale parameter drops out. The RG equation is satisfied with

$$\beta_{CS} = 0. \tag{3.70}$$

Of course, strictly speaking, β_{CS} is left undetermined by Eq. (3.65) and Eq. (3.66). But $\beta_{CS} = 0$ also is the right answer if we consider the more general RG equation satisfied by the truncated two point function itself:

$$(\partial_t + \beta_{CS}(g(t))\,\partial_g - (D-2))\, G_\beta(e^t i, e^t k) = 0. \tag{3.71}$$

We conclude that the model is thus also critical for $D > 2$; the reason is of course the presence of the Goldstone-like mode.

3.4. Gibbs States and Phase Structure

In this subsection we will discuss the set of Gibbs states of our model for the semi-infinite chain, obtained by taking the thermodynamic limit of finite chains with various b.c. As before, we will use free b.c. at 1 ($a = 1$), but the following should be noted:

Remark. If a Gibbs state $\langle \cdot \rangle$ is $O(N)$ invariant, we can modify it by introducing an arbitrary measure for a particular spin, without affecting the expectation values of $O(N)$ invariant observables. In particular these expectation values become independent of the measure ψ chosen for the b.c. at 1.

Gibbs (=equilibrium) states can be characterized by the DLR equations.[6,7] Consider a local observable $\mathcal{O}(\mathbf{s}(i),..., \mathbf{s}(k))$, i.e., a continuous function of a finite number of spins ($1 \leqslant i < k$); then the DLR equations imply that for any integer $r \geqslant 0$

$$\langle \mathcal{O} \rangle = \langle \tilde{\mathcal{O}}_r \rangle \tag{3.72}$$

where $\tilde{\mathcal{O}}_r$ is a function of $s(i)$ and $s(k+r)$

$$\tilde{\mathcal{O}}_r(\mathbf{s}, \mathbf{s}') = \frac{1}{Z} \int \mathcal{O}(\mathbf{s}(i),..., \mathbf{s}(k)) \prod_{j=i}^{k+r-1} \mathcal{T}_j\,(\mathbf{s}(j), s(j+1)) \prod_{j=i}^{k+r-1} dv(\mathbf{s}(j)) \tag{3.73}$$

with

$$Z = \int \prod_{j=i}^{k+r-1} \mathcal{T}_j\,(\mathbf{s}(j), s(j+1)) \prod_{j=i}^{k+r-1} dv(\mathbf{s}(j)) \tag{3.74}$$

So among other things, going from $\mathcal{O}$ to $\tilde{\mathcal{O}}_r$ attaches a product of r transfer matrices to the observable. We can now ask what happens if we send $r \to \infty$. In low dimensions ($1 \leqslant D \leqslant 2$) that product converges in the strong topology to $\mathcal{P}_0$, as we saw. This is already sufficient (by letting the adjoint of those operators act "to the left") to conclude that for any Gibbs state and $1 \leqslant D \leqslant 2$

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \tilde{\mathcal{P}}_0 \rangle \tag{3.75}$$

where we use same symbol $\mathcal{P}_0$ also for the integral kernel of the operator, i.e.,

$$\mathcal{P}_0(\mathbf{s}, \mathbf{s}') = \psi_0(\mathbf{s})\, \psi_0(\mathbf{s}') \equiv 1. \tag{3.76}$$

The insertion of $\mathscr{P}_0$ has the same effect as free b.c. at a point m with $m > k$, so that we conclude

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{\text{free}}. \tag{3.77}$$

So remembering that we made the general assumption $a = 1$ in this section, we have obtained the result:

Theorem 3.5. For $1 \leqslant D \leqslant 2$ with free b.c. at 1, there is a unique Gibbs state obtained as a thermodynamic limit. In particular this Gibbs state is $O(N)$ invariant.

Remark. As noted above, this implies that for $O(N)$ invariant observables, all b.c. at $i = 1$ are equivalent.

Now we turn to the cae $D > 2$. This turns out to be a little trickier and we do not obtain a complete rigorous description of all the Gibbs states. There is, however, a very natural conjecture.

Let us first state what we can prove:

Theorem 3.6. For $D > 2$ there is an uncountably infinite set of Gibbs states parametrized by the set of probability measures on the sphere S_{N-1}. The extremal states in this set are given by the probability measures δ_e concentrated in one point of $\mathbf{e} \in S_{N-1}$.

Proof. It follows from Proposition 3.1, as in the proof of Theorem 3.3, that the thermodynamic limit with a boundary measure $\tilde{\psi}$ exists for any local observable. Theorem 3.3 says first of all that the one point function in general will be different for different choices of $\tilde{\psi}$. We can easily generalize the Dirichlet b.c. discussed there by choosing

$$\tilde{\psi} = \delta_{\mathbf{e}} \tag{3.78}$$

where $\delta_{\mathbf{e}}$ is the delta function concentrated on a general point $\mathbf{e} \in S_{N-1}$. The one point function will then be

$$\langle \mathbf{s}(i) \rangle_{\infty, \mathbf{e}} = \Lambda_1(i, \infty)\, \mathbf{e} \tag{3.79}$$

so all these states are different for different choices of $\mathbf{e}$.

It is also easy to see that these generalized Dirichlet states are all extremal in the set of states given by boundary measures $\tilde{\psi}$, because they satisfy

$$\langle \mathbf{s}(\infty) \cdot \mathbf{e} \rangle_{\infty, \mathbf{e}} = 1 \tag{3.80}$$

whereas in any other state one has

$$\langle \mathbf{s}(\infty)\cdot\mathbf{e}\rangle < 1. \quad \blacksquare \tag{3.81}$$

What we do not know rigorously is whether we have exhausted all the Gibbs states by our b.c. $\tilde{\psi}$, but it is very plausible that this is so. So we state

Conjecture 3.7. For $D>2$ the set of Gibbs states is given by the set of measures $\tilde{\psi}$ on the sphere S_{N-1}.

4. PERTURBATION THEORY

4.1. Preliminaries

Perturbation Theory (PT) is nothing but the application of Laplace's method to the Gibbs measure. For a finite system, the Gibbs factor is very sharply peaked around the ground state configuration(s). To make PT work, we need a unique such ground state, and we achieve that by a suitable choice of b.c. The Gibbs measure is then to lowest order ("tree level") approximated by a Gaussian centered at the ground state, and a sequence of corrections to the Gaussian arises naturally by Laplace's method.

For a finite system one can show easily that the resulting expansion in inverse powers of β is asymptotic to the true expectation values. The usual formal PT procedure takes the thermodynamic limit term by term and hopes, if that limit exists, to obtain an expansion that is asymptotic to the infinite volume Gibbs state. It is well known that this hope fails in some cases (see refs. 4, 5, and 11, and here we will find that it fails in general for dimension $D\leqslant 2$.

From now on we assume that we have b.c. characterized by $b>0$ and $a\geqslant 0$. Then the ground state configuration is unique and is described by $\mathbf{s}(j)=\mathbf{e}_N$ for all j. For large β with high probability the spins will fluctuate not very far from $\mathbf{e}_N$. This motivates the introduction of Cartesian coordinates on the sphere S_{N-1} (as in the classic paper of Brézin and Zinn-Justin[18]) to describe these fluctuations:

$$\mathbf{s}(j)=\begin{pmatrix}\beta^{-\frac{1}{2}}\pi_j\\ \sigma_j\end{pmatrix},\qquad \text{with}\quad \sigma_j:=\pm(1-\beta^{-1}\pi_j^2)^{\frac{1}{2}}. \tag{4.1}$$

These coordinates are singular at the equator, but since we are interested in an asymptotic expansion in powers of $1/\beta$, we can ignore this fact. We can actually limit ourselves to integrating over the upper hemisphere, i.e., always choose the $+$ sign in the definition of σ_j. Finally we can extend the integration over each π_j to all of $\mathbb{R}^{N-1}$. All these changes have only an exponentially small effect (in β) on the integrals, as long as we work with a finite system $L < \infty$, and therefore they do not affect PT.

The partition function can therefore be replaced by

$$\begin{aligned} Z_L(h, a, b, D) &= \int \exp\left[\sum_{j=1}^{L-1} b_j\,(\boldsymbol{\pi}_j \cdot \boldsymbol{\pi}_{j+1} + \beta(1-\beta^{-1}\boldsymbol{\pi}_j^2)^{\frac{1}{2}}\,(1-\beta^{-1}\boldsymbol{\pi}_{j+1}^2)^{\frac{1}{2}}) \right.\\ &\quad \left. +\beta h \sum_{j=1}^{L} g_j(1-\beta^{-1}\boldsymbol{\pi}_j^2)^{\frac{1}{2}} + \beta a\,(1-\beta^{-1}\boldsymbol{\pi}_1^2)^{\frac{1}{2}} + \beta b(1-\beta^{-1}\boldsymbol{\pi}_L^2)^{\frac{1}{2}}\right] \\ &\quad \times \exp\left[-\frac{1}{2}\sum_{j=1}^{L} \ln(1-\beta^{-1}\boldsymbol{\pi}_j^2)\right] \prod_{j=1}^{L} d\boldsymbol{\pi}_j. \end{aligned} \tag{4.2}$$

and accordingly for the expectation values. In other words, for the purpose of PT we are reduced to studying the Gibbs measures

$$d\mu_L(\boldsymbol{\pi}_1,..., \boldsymbol{\pi}_L) = \frac{1}{Z_L} \exp(A_L) \prod_{j=1}^{L} d\boldsymbol{\pi}_j \tag{4.3}$$

with

$$\begin{aligned} A_L &= \left[\sum_{j=1}^{L-1} b_j\,(\boldsymbol{\pi}_j \cdot \boldsymbol{\pi}_{j+1} + \beta(1-\beta^{-1}\boldsymbol{\pi}_j^2)^{\frac{1}{2}}\,(1-\beta^{-1}\boldsymbol{\pi}_{j+1}^2)^{\frac{1}{2}}) \right.\\ &\quad \left. +\beta h \sum_{j=1}^{L} g_j(1-\beta^{-1}\boldsymbol{\pi}_j^2)^{\frac{1}{2}} + \beta a(1-\beta^{-1}\boldsymbol{\pi}_1^2)^{\frac{1}{2}} + \beta b(1-\beta^{-1}\boldsymbol{\pi}_L^2)^{\frac{1}{2}}\right] \\ &\quad -\frac{1}{2}\sum_{j=1}^{L} \ln(1-\beta^{-1}\boldsymbol{\pi}_j^2). \end{aligned} \tag{4.4}$$

It is now clear how to proceed; we expand A in powers of β; in this paper we will be content to do this to order $1/\beta$:

$$A = 2^{D-1}\beta V_L(1+h) + a\beta b_1 + b\beta + A^{(0)} + A^{(1)} + O(\beta^{-2}) \tag{4.5}$$

where the piece $O(\beta^0)$ is quadratic in the $\boldsymbol{\pi}_j$ variables:

$$A^{(0)} = -\frac{1}{2}\sum_{j=1}^{L-1} b_j(\boldsymbol{\pi}_{j+1} - \boldsymbol{\pi}_j)^2 - \frac{h}{2}\sum_{j=1}^{L} g_j \boldsymbol{\pi}_j^2 - \frac{a}{2}\boldsymbol{\pi}_1^2 - \frac{b}{2}\boldsymbol{\pi}_L^2 \tag{4.6}$$

and the term $O(\beta^{-1})$ is

$$A^{(1)} = -\frac{1}{2\beta}\left[\sum_{j=1}^{L-1} b_j \left(\frac{\pi_{j+1}^2 - \pi_j^2}{2}\right)^2 + \frac{h}{4}\sum_{j=1}^{L} g_j(\pi_j^2)^2 - \sum_{j=1}^{L} \pi_j^2 + \frac{1}{4}a(\pi_1^2)^2 + \frac{1}{4}b(\pi_L^2)^2\right]. \quad (4.7)$$

As long as not $a=b=h=0$, $-A^{(0)}$ is a positive definite quadratic form in the π_j which we write as

$$A^{(0)} \equiv -\tfrac{1}{2}(\pi, Q\pi) \quad (4.8)$$

where π stands for the vector $(\pi_i^a)_{i=1,\dots,L}^{a=1,\dots,N-1}$ in $\mathbb{R}^{(N-1)L}$ and Q has matrix elements $Q_{ik}^{ab} = \delta^{ab} q_{ik}$. We combine $A^{(0)}$ with the Legesgue measure to produce the Gaussian probability measure

$$d\mu_C(\pi) = \frac{1}{Z_L^{(0)}} \exp(-\tfrac{1}{2}(\pi, Q\pi) \quad (4.9)$$

with covariance $C = Q^{-1}$. C is proportional to the identity operator in the internal space $\mathbb{R}^{N-1}$, i.e., its matrix elements are of the form $C_{ik}^{ab} = \delta^{ab} c_{ik}$. $A^{(1)}$, even though it contains also terms quadratic in π, is treated as an interaction because it is of order β^{-1}.

4.2. Spin Two Point Function

We first derive the explicit form of PT up to order β^{-2} for the invariant spin-spin correlation in terms of the covariance of the Gaussian measure up to order β^{-2}. First note that

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(k)\rangle = 1 - \frac{1}{2\beta}\langle(\pi_i - \pi_k)^2\rangle - \frac{1}{8\beta^2}\langle(\pi_k^2 - \pi_k^2)^2\rangle + O(\beta^{-2}) \quad (4.10)$$

provided we have $a, b, h \geqslant 0$ and at least one of them > 0, because then the ground state will have all spins aligned parallel to $\mathbf{e}_N$. As usual, PT is generated by expanding also the interaction in inverse powers of β in the Gibbs measure; as is well known, the correct normalization leads to the phenomenon that "vacuum graphs cancel," i.e., only terms survive in which the interaction is connected by "lines" (covariances) to the observable. Thus

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(k)\rangle = 1-\frac{1}{2\beta}\langle(\pi_i-\pi_k)^2\rangle_c-\frac{1}{8\beta^2}\langle(\pi_i^2-\pi_k^2)^2\rangle_c - \left\langle \frac{1}{2\beta}(\pi_i-\pi_k)^2;\, A^{(1)}\right\rangle_c + O(\beta^{-3}) \tag{4.11}$$

where the semicolon indicates that only contributions are to be taken that connect the expression to the left with that to the right of it. $\langle . \rangle_c$ denotes the Gaussian expectation value with covariance c and $A^{(1)}$ is defined in Eq. (4.7).

The term order β^{-1} is therefore

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(k)\rangle^{(1)} = -\frac{N-1}{2\beta}(c_{ii}+c_{kk}-2c_{ik}). \tag{4.12}$$

To order β^{-2} we find

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(k)\rangle^{(2)} = (\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V}) \tag{4.13}$$

with

$$(\mathrm{I}) \equiv -\frac{1}{8\beta^2}\langle(\pi_i^2-\pi_k^2)^2\rangle_c = -\frac{N-1}{8\beta^2}[(c_{ii}^2+c_{kk}^2)(N+1)-4c_{ik}^2-2(N-1)\,c_{ii}c_{kk}], \tag{4.14}$$

$$(\mathrm{II}) \equiv -\frac{1}{4\beta^2}\sum_{j=1}^{L}\langle(\pi_i-\pi_k)^2;\,\pi_j^2\rangle_c = -\frac{N-1}{2\beta^2}\sum_{j=1}^{L}(c_{ij}-c_{kj})^2, \tag{4.15}$$

$$\begin{aligned}(\mathrm{III}) &\equiv \frac{1}{16\beta^2}\sum_{j=1}^{L-1} b_j\langle(\pi_i-\pi_k)^2;\,(\pi_j^2-\pi_{j+1}^2)^2\rangle_c \\ &= \frac{N-1}{4\beta^2}\sum_{j=1}^{L-1} b_j\{(N+1)[(c_{jj}(c_{ij}-c_{kj})^2+c_{j+1,\,j+1}(c_{i,\,j+1}-c_{k,\,j+1})^2] \\ &\quad -(N-1)[c_{jj}(c_{i,\,j+1}-c_{k,\,j+1})^2+c_{j+1,\,j+1}(c_{ij}-c_{kj})^2] \\ &\quad -4c_{j,\,j+1}(c_{ij}-c_{kj})(c_{i,\,j+1}-c_{k,\,j+1})\},\end{aligned} \tag{4.16}$$

$$(\mathrm{IV}) \equiv \frac{h}{8\beta^2}\sum_{j=1}^{L} g_j\langle(\pi_i-\pi_k)^2;\,(\pi_j)^2\rangle = \frac{(N-1)(N+1)}{4\beta^2}h\sum_{j=1}^{L} g_j c_{jj}(c_{ij}-c_{kj})^2 \tag{4.17}$$

and finally the boundary contribution

$$(\mathrm{V}) \equiv \frac{1}{8\beta^2} \langle (\pi_i - \pi_k)^2; a(\pi_1^2)^2 + b(\pi_L^2)^2 \rangle$$
$$= \frac{(N-1)(N+1)}{8\beta^2} [a c_{11} (c_{i1} - c_{k1})^2 + b c_{LL} (c_{iL} - c_{kL})^2]. \tag{4.18}$$

It is advantageous to rewrite contribution (III) as follows:

$$(\mathrm{III}) = (\mathrm{IIIa}) + (\mathrm{IIIb}) + (\mathrm{IIIc}) \tag{4.19}$$

with

$$(\mathrm{IIIa}) \equiv \frac{(N-1)\,N}{4\beta^2} \sum_{j=1}^{L-1} b_j (c_{jj} - c_{j+1,\,j+1}) [(c_{ij} - c_{kj})^2 - (c_{i,\,j+1} - c_{k,\,j+1})^2] \tag{4.20}$$

$$(\mathrm{IIIb}) \equiv \frac{N-1}{4\beta^2} \sum_{j=1}^{L-1} b_j (c_{jj} + c_{j+1,\,j+1}) (c_{ij} - c_{kj} - c_{i,\,j+1} + c_{k,\,j+1})^2 \tag{4.21}$$

$$(\mathrm{IIIc}) \equiv \frac{N-1}{2\beta^2} \sum_{j=1}^{L-1} b_j (c_{jj} + c_{j+1,\,j+1} - 2c_{j,\,j+1}) (c_{ij} - c_{kj}) ((c_{i,\,j+1} - c_{k,\,j+1}). \tag{4.22}$$

4.3. The Covariance Without Magnetic Field

To proceed, we have to compute the covariance C more explicitly. We only do this for the simplest case of free b.c. at 1 and no magnetic field, i.e., $b = h = 0$, but with $a > 0$. We can ignore the internal space in this computation, i.e., put $N = 2$. Then, with $\mathbf{x} \in \mathbb{R}^L$, we have

$$(\mathbf{x}, Q\mathbf{x}) = \sum_{j=1}^{L-1} b_j (x_{j+1} - x_j)^2 + a x_1^2. \tag{4.23}$$

For convenience we define in the following $b_0 \equiv a$ and $b_L \equiv b$. Q is already given in the form

$$Q = L^T L \tag{4.24}$$

where L is a lower triangular (actually bidiagonal) matrix with elements

$$l_{kk} = \sqrt{b_{k-1}} \qquad \text{and}$$
$$l_{k+1,\,k} = -\sqrt{b_k} \qquad \text{for} \quad k = 1, 2,..., L \tag{4.25}$$

all other elements being zero. We now split

$$L = D + N \tag{4.26}$$

where $D = \mathrm{diag}(\sqrt{b}_0, \ldots, \sqrt{b_{L-1}})$ and N is nilpotent. Then

$$C = Q^{-1} = (\mathbb{1} + D^{-1}N)^{-1} D^{-2} (\mathbb{1} + N^T D^{-1})^{-1} \tag{4.27}$$

$D^{-1}N$ has only nonzero matrix elements d_{ik} for $i = k+1$, and they are all equal to -1. Therefore $Y \equiv (\mathbb{1} + D^{-1}N)^{-1}$ has the matrix elements

$$\begin{aligned} y_{ik} &= 1 \qquad \text{for} \quad i \geqslant k \\ y_{ik} &= 0 \qquad \text{otherwise.} \end{aligned} \tag{4.28}$$

So, using the shorthand $m_{ik} \equiv \min(i, k)$ we obtain for the covariance

$$c_{ik} = \sum_{j=1}^{L} b_{j-1}^{-1} y_{ij} y_{kj} = \sum_{r=0}^{m_{ik}-1} b_j^{-1} \tag{4.29}$$

for $b \equiv b_L = 0$. If we introduce the further shorthand

$$B_{ik} \equiv \sum_{j=i}^{k-1} b_j^{-1} \tag{4.30}$$

we can write the covariance we found as

$$c_{ik}(a, 0) = B_{0m_{ik}} = B_{1m_{ik}} + a^{-1} \tag{4.31}$$

where the second argument is reserved for the parameter b.

The covariance for general values of a and b (still without a magnetic field) is obtained from this by realizing that changing the b.c. at L is a rank one perturbation:

$$c_{ik}(a, b) = c_{ik}(a, 0) - (b^{-1} + c_{LL}(a, 0))^{-1} c_{iL}(a, 0)\, c_{kL}(a, 0). \tag{4.32}$$

This can be written in more compact form if we use the definitions $M_{ik} \equiv \max(i, k)$ and

$$B \equiv (a^{-1} + b^{-1} + B_{1L})^{-1} = \frac{1}{B_{0,\,L+1}}: \tag{4.33}$$

as

$$c_{ik}(a, b) = BB_{0m_{ik}} B_{M_{ik}, L+1}. \tag{4.34}$$

We note some special cases:

$$c_{ik}(0, b) = B_{M_{ik}L} + b^{-1} = B_{M_{ik}, L+1} \tag{4.35}$$

$$c_{ik}(\infty, b) = BB_{1m_{ik}} B_{M_{ik}, L+1} \tag{4.36}$$

$$c_{ik}(a, \infty) = BB_{M_{ik}L} B_{0m_{ik}}. \tag{4.37}$$

We also note the following combination of covariances:

$$c_{ii}(a, b) + c_{kk}(a, b) - 2c_{ik}(a, b) = B_{ik}(1 - BB_{ik}). \tag{4.38}$$

The behavior for large L and the asymptotic behavior of the infinite volume covariances for $k \to \infty$ can be obtained from Proposition 2.1, which implied that for $1 \leqslant D < 2$ $B_{ik} = O(k^{2-D})$, for $D = 2$ $B_{ik} = O(\ln(k))$ and for $D > 2$ $B_{ik} = O(1)$. The thermodynamic limit is particularly easy to take for $b = 0$, because in that case by Eq. (4.31) the covariance does not show any dependence on L.

In general for $1 \leqslant D \leqslant 2$

$$\lim_{L \to \infty} c_{ik}(a, b) = B_{0m_{ik}} \tag{4.39}$$

independently of b, provided $a > 0$. For $1 \leqslant D \leqslant 2$ and $a = 0$ the thermodynamic limit of the covariance does not exist.

For $D > 2$ the thermodynamic limit always exists, but it depends on the b.c. parameter b.

For the general case with a magnetic field there are no such simple expressions for the covariance. A very detailed discussion with many explicit and lengthy formulae can be found in ref. 10.

4.4. Explicit Evaluation to One Loop

We will first discuss the simplest case of the "energy," that is the two point function of two neighboring spins.

To order β^{-1} ("tree level"), we have

$$\langle \mathbf{s}(i) \cdot \mathbf{s}(i+1) \rangle = 1 - \frac{N-1}{2\beta} (c_{ii} + c_{i+1, i+1} - 2c_{i, i+1}) + O(\beta^{-2}). \tag{4.40}$$

Using Eq. (4.38) we thus find

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(i+1)\rangle = 1-\frac{N-1}{2\beta_i}(1-Bb_i^{-1})+O(\beta^{-2}) \tag{4.41}$$

and taking the thermodynamic limit of the first order term we obtain

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(i+1)\rangle^{(1)} = b_i^{-1} \qquad (1\leqslant D\leqslant 2) \tag{4.42}$$

and

$$\langle \mathbf{s}(i)\cdot\mathbf{s}(i+1)\rangle^{(1)} = b_i^{-1}\left(1-\frac{b_i^{-1}}{b^{-1}+B_{0\infty}}\right) \qquad (D>2). \tag{4.43}$$

This means that for $D>2$ already at tree level PT shows a dependence on b.c. even for such a simple $O(N)$ invariant observable. By expanding the nonperturbative solution of the model, analyzed in the previous section, one can see that this is a real effect, related to the occurrence of spontaneous symmetry breaking (SSB). The dependence on the b.c. parameter b drops out, however, if we use free b.c. at 1 ($a=0$). This happens because for an invariant observable we can always introduce an arbitrary probability measure for one and only one spin, without any effect, as explained earlier.

To the next order ("one loop level") the computation becomes a little more involved. Since for $D>2$ already the tree level term depends on the b.c., we assume from now on $1\leqslant D\leqslant 2$.

We first consider the case $a>0$, $b=0$. It is easy to see that in this case term (V) vanishes in the thermodynamic limit. Term (IV) is absent because we put the magnetic field h equal to zero. For the other terms, after some trivial algebra one obtains

$$(\mathrm{I}) = -\frac{N-1}{8\beta_i^2}\,[N+1+4b_iB_{0i}], \tag{4.44}$$

$$(\mathrm{II}) = -\frac{N-1}{8\beta_i^2}\,[4(L-i)], \tag{4.45}$$

$$(\mathrm{III}) = \frac{N-1}{8\beta_i^2}\,[2(N-1)+4(L-i)+4b_iB_{0i}], \tag{4.46}$$

which adds up to

$$\langle \mathbf{s}(i) \cdot \mathbf{s}(i+1) \rangle^{(2)} (a, 0) = \frac{(N-1)(N-3)}{8\beta_i^2}. \tag{4.47}$$

The case $h = 0$ with general b.c. can also be worked out explicitly. First note that the term (V) still does not contribute in the thermodynamic limit: we have

$$(\mathrm{V}) = \frac{(N-1)(N+1)}{8\beta^2} \left[a c_{11} (c_{i1} - c_{k1})^2 + b c_{LL} (c_{iL} - c_{kL})^2 \right]. \tag{4.48}$$

For $a > 0$ and any b the first term goes to 0 as $L \to \infty$, since by Eq. (4.39) $\lim_{L \to \infty} (c_{iL} - c_{kL}) = 0$. For the second term we notice

$$b c_{LL} (c_{iL} - c_{kL})^2 = b B^3 B_{0m} B_{m,\,L+1} (B_{0i} - B_{0k})^2 \, b^{-2} = b B^3 B_{0m} B_{m,\,L+1} B_{ik}^2 \to 0 \tag{4.49}$$

for $L \to \infty$.

Inserting our formula Eq. (4.34) into the expressions at the end of Subsection 4.2, the result for (II) and (III) splits into sums from 1 to $i-1$ and from i to $L-1$ or L. For the finite sums the limit $L \to \infty$ can be taken termwise and is actually zero, because each propagator carries a factor B which goes to 0 as $L \to \infty$. After some algebra we are left with

$$(\mathrm{I}) = -\frac{N-1}{8\beta_i^2} \left[N + 1 + 4 b_i B_{0i} \right], \tag{4.50}$$

$$(\mathrm{II}) = -\frac{N-1}{8\beta_i^2} B^2 \sum_{j=i+1}^{L} B_{j,\,L+1}^2 \tag{4.51}$$

$$(\mathrm{IIIa}) \sim \frac{(N-1)\, N}{4\beta^2} \left\{ b_i^{-2} + B^2 B_{ik}^2 \sum_{j=i+1}^{L-1} \left[2 b_j^{-1} B_{j,\,L+1} - 4 B B_{j,\,L+1}^2 \right] \right\} \tag{4.52}$$

$$(\mathrm{IIIb}) \sim \frac{N-1}{4\beta^2} \left\{ b_i^{-2} - B^3 B_{ik}^2 \sum_{j=i+1}^{L-1} 2 b_j^{-1} B_{j,\,L+1}^2) \right\} \tag{4.53}$$

$$(\mathrm{IIIc}) \sim \frac{N-1}{2\beta^2} B_{ik}^2 \sum_{j=i+1}^{L-1} \left[B^3 B_{j,\,L+1}^2 - B^2 b_j^{-1} B_{j,\,L+1} - B^3 b_j^{-1} B_{j,\,L+1}^2 \right]. \tag{4.54}$$

Here the symbol $\sim$ means equality up to terms vanishing in the limit $L \to \infty$. In arriving at these expressions we used the fact that some terms vanish in the limit $L \to \infty$:

Lemma 4.1. For $1 \leqslant D \leqslant 2$ the following holds in the limit $L \to \infty$:

(1)
$$B^3 \sum_{j=i+1}^{L-1} b_j^{-2} B_{j,\,L+1} \to 0, \tag{4.55}$$

(2)
$$B^3 \sum_{j=i+1}^{L-1} b_j^{-2} B_{0j} \to 0, \tag{4.56}$$

(3)
$$B^3 \sum_{j=i+1}^{L-1} b_j^{-3} \to 0. \tag{4.57}$$

Proof.

$$0 < (1) \leqslant B^2 \sum_{j=i+1}^{L-1} b_j^{-2}$$

$$0 < (2) \leqslant B^2 \sum_{j=i+1}^{L-1} b_j^{-2}.$$

Using Proposition 2.1, it is seen easily that both upper bounds go to zero for $L \to \infty$. (3) is even more elementary. ∎

There are also sums that converge to nonzero limits:

Lemma 4.2. For $1 \leqslant D \leqslant 2$ and $b > 0$

(1)
$$I_1 \equiv \lim_{L \to \infty} B^2 \sum_{j=i+1}^{L-1} b_j^{-1} B_{j,\,L+1} = \tfrac{1}{2} \tag{4.58}$$

(2)
$$I_2 \equiv \lim_{L \to \infty} B^3 \sum_{j=i+1}^{L-1} b_j^{-1} B_{j,\,L+1}^2 = \tfrac{1}{3}. \tag{4.59}$$

For $b = 0$ I_1 and I_2 vanish, because each term in the sums vanishes before taking the limit.

Proof. The two statements are proven by "summation by parts." First note that for any sequences f_i, g_i we have

$$\sum_{j=A}^{B} [f_{j+1} - f_j]\, g_j = -\sum_{j=A}^{B} [g_{j+1} - g_j]\, f_{j+1} + f_{B+1} g_{B+1} - f_A g_A. \tag{4.60}$$

Applying this to the sum in (1) above and noting that $b_j^{-1} = B_{j+1, L+1} - B_{j, L+1}$, we obtain

$$\sum_{j=i+1}^{L-1} b_j^{-1} B_{j, L+1} = \sum_{j=i+1}^{L-1} b_j^{-2} + \tfrac{1}{2} B_{i+1, L+1}^2 - \tfrac{1}{2} b_L^{-2}. \tag{4.61}$$

From this the assertion (1) follows easily, using Lemma 4.1.

Applying the formula (4.60) to the expression in (2) above yields, after some simplification,

$$\sum_{j=i+1}^{L-1} b_j^{-1} B_{j, L+1}^2 = \sum_{j=i+1}^{L-1} b_j^{-2} B_{j, L+1} - \tfrac{1}{3} \sum_{j=i+1}^{L-1} b_j^{-3} - \tfrac{1}{3} b_L^{-3} + \tfrac{1}{3} B_{i+1, L+1}^3. \tag{4.62}$$

Again the assertion follows from Lemma 4.1. ∎

Now we can add up all the contributions; the divergent terms cancel and we obtain

$$(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \sim \frac{N-1}{8\beta_i^2} \{-4B^2 B_{L, L+1}^2 + (N+1) - 8(N+1) I_1 + 4N I_2\}. \tag{4.63}$$

For $b = 0$ I_1 and I_2 vanish, $B^2 B_{L, L+1}^2 = 1$ and we recover our old result (4.47). For $b > 0$, however, the term $B^2 B_{L, L+1}^2$ vanishes and the sums I_1 and I_2 converge to $1/2$ and $1/3$. So we obtain

$$\langle \mathbf{s}(i) \cdot \mathbf{s}(i+1) \rangle^{(2)} (a, b) = \frac{(N-1)(N-5)}{24\beta_i^2} \tag{4.64}$$

for $b > 0$.

So we have found that in our model PT gives results depending on b.c. for any dimension (except for $N = 2$)! While this is a sensible result for $D > 2$ due to the occurrence of SSB, it signals a disease of PT for $1 \leqslant D \leqslant 2$, where we have a unique Gibbs state. The right asymptotic expansion to order β^{-2} is obtained only with free b.c., where the thermodynamic limit is reached already for finite L. Free b.c. are "DLR b.c." for our model, corresponding to exactly integrating out the variables outside our box.

We now turn to the general case $\langle \mathbf{s}(i) \cdot \mathbf{s}(k) \rangle^{(2)} (a, b)$ for $i > k$. The computation is similar, only this time we have to split the sums into 3 parts: from 1 to $i-1$, from i to $k-1$ and from k to L or $L-1$. For the sums of finite range we can again take the termwise limit $L \to \infty$, which makes the sums from 1 to $i-1$ disappear. The sums from i to $k-1$ produce, among others

$$\sum_{j=i}^{k-1} b_j^{-1} B_{ij} = -\tfrac{1}{2} \sum_{j=i}^{k-1} b_j^{-2} + \tfrac{1}{2} B_{ik}^2. \tag{4.65}$$

Using this formula is is not difficult to find the final result:

Proposition 4.3. The one loop PT result for the spin-spin two point function is for free b.c. ($b=0$):

$$\langle \mathbf{s}(i) \cdot \mathbf{s}(k) \rangle^{(2)} (a, b) = \frac{(N-1)(N-3)}{8\beta^2} B_{ik}^2 - 2(N-1) \left[\sum_{j=i}^{j=k-1} b_j^{-2} - B_{ik}^2 \right] \tag{4.66}$$

and represents the correct aymptotics in the thermodynamic limit. Taking the thermodynamic limit termwise for $b > 0$, we instead obtain the incorrect result

$$\langle \mathbf{s}(i) \cdot \mathbf{s}(k) \rangle^{(2)} (a, b) = \frac{(N-1)(N-5)}{8\beta^2} B_{ik}^2 - 2(N-1) \left[\sum_{j=i}^{j=k-1} b_j^{-2} - B_{ik}^2 \right]. \tag{4.67}$$

A final remark concerns free b.c. at site 1 ($a=0$): since the result for $b=0$ is strictly independent of L as well as a, and since the labeling of the sites can be inverted ($j \to L-j$), the result for $a>0$, $b=0$ is identical with the result for $a=0$, $b>0$, and therefore produces the correct asymptotics.

Comparing the results obtained with different b.c, we can sum up what we have found as follows: the second order PT term for the invariant two point function, computed by the conventional termwise thermodynamic limit with b.c. parameters $a, b > 0$ differs from the correct second order term by

$$\delta \langle \mathbf{s}(i) \cdot \mathbf{s}(k) \rangle^{(2)} (a, b) = \frac{(N-1)(N-2)}{12\beta^2} B_{ik}^2 . \tag{4.68}$$

This dependence on b.c. we have found highlights a general problem of PT: an infrared regulator is needed, but the standard procedure of removing that regulator termwise yields ambiguous results. In general they are incorrect, and only in our simplified model we are in the lucky situation to know the DLR b.c. which yield the true answer.

The case $N = 2$ is special: we have not found manifest signs of a disease of PT. In fact for the translation invariant $O(2)$ model it has been *proven* in ref. 19 that standard PT yields the correct asymptotic expansion.

4.5. Remarks on the Magnetic Field as Infrared Regulator

A popular infrared regulator in the non-linear $O(N)$ σ-models is the magnetic field. It was used for the first detailed study of the perturbative Renormalization Group and the asymptotic freedom predicted by it.[18] On the other hand it has been known for many years that the usual procedure of doing the perturbation expansion in the presence of a magnetic field and than removing it termwise yields incorrect results already for a finite number of spins[20] and in $D = 1$.[4] Since our models are essentially one-dimensional, the same phenomenon is expected to occur also here. This is discussed in detail in ref. 10.

Here it would lead us too far afield to go into this very technical matter, which involves interesting methods from the theory of continued fractions. But it should be seen that introducing the b.c. parameters a and b essentially amounts to the introduction of a *local* magnetic field, and we have seen in the previous subsection that for $N > 2$ with $a > 0$ and $b > 0$ PT produces incorrect results, independent of the values of a and b; hence sending $a, b \to 0$ in the end does not help. So it should be clear that one cannot hope for anything better with a global magnetic field.

5. PERCOLATION PROPERTIES

In this section we discuss briefly some percolation properties of the model. Even though percolation is rather trivial in our essentially one-dimensional systems, we find it worthwhile to check whether the general ideas of refs. 2, 8, and 9 apply in this case. Not surprisingly, we find again the familiar dichotomy between the situation in $D \leqslant 2$ and $D > 2$.

We are interested in the percolation properties of sets defined by the spin $\mathbf{s}$ lying in certain open connected subsets A of the sphere S_{N-1}, such as the "polar caps" $\mathscr{P}_\epsilon^+ \equiv \mathbf{s} \cdot \mathbf{e}_N > \epsilon/2$ and "equatorial strips" $\mathscr{S}_\epsilon \equiv |\mathbf{s} \cdot \mathbf{e}_N| < \epsilon/2$ discussed in refs. 2, 8, and 9. For simplicity we say "a certain subset $A \subset S_{N-1}$ percolates" when we mean that the set of points of our lattice $\{i \in \mathbb{Z}^D \mid \mathbf{s}(i) \in A\}$ percolates.

Let us first discuss the case of no symmetry breaking ($1 \leqslant D \leqslant 2$):

Theorem 5.1. For $1 \leqslant D \leqslant 2$ a subset $A \subset S_{N-1}$ whose complement is open and nonempty never percolates.

Remark. If one interpretes the model as living on $\mathbb{Z}^D$, this can be viewed for $D=1$ as the formation of "islands" and for $D=2$ as the "ring formation" discussed in refs. 2, 8, and 9.

Proof. Assume the contrary. Consider the characteristic function $\chi_A(s(i))$. Then by Theorem 3.2

$$\langle \chi_A(s(i)) \rangle = \int_A d\nu(\mathbf{s}), \tag{5.1}$$

which is a number independent of i and <1. On the other hand, if A percolates,

$$\lim_{i\to\infty} \langle \chi_A(s(i)) \rangle \geqslant \langle \lim_{i\to\infty} \chi_A(s(i)) \rangle = 1 \tag{5.2}$$

by Fatou's lemma, which is a contradiction. ∎

We now turn to the case of spontaneous symmetry breaking ($D>2$). In this case it is to be expected that in the Gibbs state $\langle . \rangle_{\infty,\,\mathrm{Dir}}$, obtained as the thermodynamic limit with $\mathbf{e}_N$-Dirichlet b.c., there is percolation of any neighborhood of the "north pole" $\mathbf{e}_N$. To actually prove this requires rather detailed technical estimates (cf. ref. 10). Here we will limit ourselves to giving a simple proof of this fact for $D>3$) and then show that it can be extended to $D>5/2$.

Theorem 5.2. For $D>5/2$, in the state obtained as the thermodynamic limit with $\mathbf{e}_N$-Dirichlet b.c., any open set A containing the point $e_N \in S_{N-1}$ percolates.

Proof. We will show that the probabilities for the events

$$a_i \equiv \{\mathbf{s}(i) \notin A\} \tag{5.3}$$

are summable; the theorem then follows from the Borel–Cantelli Lemma, which states that in this case with probability 1 only finitely many of the events a_i occur.

We have the following

Proposition 5.3. Let χ_i^ϵ be the characteristic function of the set $B \equiv \{\mathbf{s}(i) \in S_{N-1} \mid |\mathbf{s}(i)-\mathbf{e}_N| > \epsilon\}$. Then

$$\langle \chi_i^\epsilon \rangle_{\infty,\,\mathrm{Dir}} < c i^{2-D}. \tag{5.4}$$

Proof. Obviously for any $n > 0$

$$\chi_i^\epsilon \leqslant \left(\frac{|\mathbf{s}(i)-\mathbf{e}_N|}{\epsilon}\right)^n. \tag{5.5}$$

Using $n = 2$ and the results of Section 3 we obtain:

$$\begin{aligned}\langle |s(i)-\mathbf{e}_N|^2\rangle_{\infty,\,\mathrm{Dir}} &= \left\langle \sum_{a=1}^{N-1} s_a(i)^2 \right\rangle_\infty = \frac{N-1}{N}\left(1-\frac{\Lambda_2(\infty)}{\Lambda_2(i)}\,y\right)\\ &\sim -\frac{l(l+N-2)}{2\beta}\,\frac{1}{2^D D(D-2)}\,i^{2-D}.\end{aligned} \tag{5.6}$$

The last expression is clearly summable over i for $D > 3$, so this proves percolation for $D > 3$.

To extend the proof to $D > 5/2$, we have to work a little more. Again by the Borel–Cantelli Lemma, the claim will follow from a sharpening of Proposition 5.3:

Proposition 5.4.

$$\langle \chi_i^\epsilon \rangle_{\mathrm{Dir}} < c i^{2(2-D)}. \tag{5.7}$$

Proof. We choose $n = 4$ in the inequality 5.5. Denoting $\mathbf{s}(i)\cdot\mathbf{e}_N \equiv z$ we expand $|s(i)-\mathbf{e}_N|^4$ in Gegenbauer polynomials in z:

$$|s(i)-\mathbf{e}_N|^4 = 1-2z^2+z^4 = a_o + a_2 C_2^{\frac{N}{2}-1}(z) + a_4 C_4^{\frac{N}{2}-1}(z) \tag{5.8}$$

with

$$a_o = \frac{N^2-1}{N(N+2)}, \tag{5.9}$$

$$a_2 = \frac{4(N+1)}{N(N-2)(N+4)}, \tag{5.10}$$

$$a_4 = \frac{24}{N(N^2-4)(N+4)}. \tag{5.11}$$

Thus we obtain, proceeding as in Section 3

$$\begin{aligned}\langle |s(i)-\mathbf{e}_N|^4\rangle_{\infty,\,\mathrm{Dir}} &= (\psi_o, |s-\mathbf{e}_N|^4\, \mathcal{T}_{i\infty}\delta_{\mathbf{e}_N})\\ &= a_o + a_2 C_2^{\frac{N}{2}-1}(1)\,\Lambda_2(i,\infty) + a_4 C_4^{\frac{N}{2}-1}(1)\,\Lambda_2(i,\infty).\end{aligned} \tag{5.12}$$

Using now the asymptotics on $\Lambda_l(i, \infty)$ found in Section 3, we find that the constant terms, as well as the terms $O(i^{2-D})$, cancel and we are left with an expression $O(i^{2(2-D)})$ as claimed. ∎

Remark. By developing the asymptotics of $\Lambda_l(i, \infty)$ further, one could presumably push the percolation threshold down to 2. Alternatively, percolation for all $D > 2$ would follow also from the fact that the kernel $\mathcal{T}_{i\infty}(\mathbf{e}_n, \mathbf{s})$ decays as a Gaussian with variance $\langle |s(i) - \mathbf{e}_N|^2 \rangle_{\infty, \mathrm{Dir}}$.

6. CONCLUSIONS

The main purpose of this study of a solvable familiy of models was to test certain ideas of Patrascioiu and Seiler concerning the two-dimensional $O(N)$ spin models. Let us review the outcome.

The central thesis of Patrascioiu and Seiler put forward in numerous publications (see for instance refs. 2, 8, and 9) and references therein) is that there is no fundamental qualitative difference between the "abelian" case $N = 2$ and the "nonabelian" one $N > 2$. This is fully confirmed in the solvable models studied here: critical behavior depends only on the dimension parameter D, with the models becoming critical at $D = 2$ independent of N.

Another point stressed by Patrascioiu and Seiler[5] is the fact that perturbation theory is ambiguous in $D \leqslant 2$ and for $N > 2$; they suggest that the difference found in the perturbative renormalization group between the cases $N = 2$ and $N > 2$ is an artefact of perturbation theory. This is again borne out here, as discussed in Section 4.

Finally concerning percolation properties, we find that the "ring formation," proposed in ref. 2, 8, and 9 for the "soft phase" in $D = 2$ actually takes place in the models studied here.

Due to their simplicity these models are, however, lacking certain features that the translation invariant $O(N)$ models possess: due to their essentially one-dimensionl nature there is no difference between "ring formation" and formation of "islands" typical for the massive high temperature phase. In fact for $D = 2$ the high temperature phase has been eliminated altogether, in accordance with the fact the models are more ordered than the translation invariant ones. So this unfortunately eliminates the possibility of studying critical behavior in β.

In our view the models studied here give support for the ideas of Patrascioiu and Seiler, but of course they cannot provide anything like a proof of them.

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